

Quasi *BPS* Wilson loops, localization of loop equation by homology and exact beta function in the large- N limit of $SU(N)$ Yang-Mills theory

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Quasi BPS Wilson loops, localization of loop equation by homology and exact beta function in the large- N limit of $SU(N)$ Yang-Mills theory

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ABSTRACT: We localize the loop equation of large- N YM theory in the anti-self-dual variables on a critical equation for an effective action by means of homological methods as opposed to the cohomological localization of equivariantly closed forms in local field theory. Our localization occurs for some special simple quasi BPS Wilson loops, that have no perimeter divergence and no cusp anomaly for backtracking cusps, in a partial Eguchi-Kawai reduction from four to two dimensions of the non-commutative theory in the limit of infinite non-commutativity and in a lattice regularization in which the anti-self-dual integration variables live at the points of the lattice, thus implying an embedding of parabolic Higgs bundles in the YM functional integral. We find that the beta function of the effective action is saturated by the non-commutative anti-self-dual vortices of the Eguchi-Kawai reduction. An exact canonical beta function of Novikov-Shifman-Vainshtein-Zakharov type, that reproduces the universal first and second perturbative coefficients follows by the localization on vortices. Finally we argue that a scheme can be found in which the canonical coupling coincides with the physical charge between static quark sources in the large- N limit and we compare our theoretical calculation with some numerical lattice result.

KEYWORDS: $1/N$ Expansion, Renormalization Group, Non-Commutative Geometry, Differential and Algebraic Geometry

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1 Introduction

In this paper we revisit and refine our result [1] on the exact beta function in the large- N limit of the pure Yang-Mills theory. From a computational point of view we can summarize our result as follows.

There exists a renormalization scheme in which the large- N canonical beta function of the pure YM theory is given by:

$$\frac{\partial g_c}{\partial \log \Lambda} = \frac{-\beta_0 g_c^3 + \frac{\beta_J}{4} g_c^3 \frac{\partial \log Z}{\partial \log \Lambda}}{1 - \beta_J g_c^2} \tag{1.1}$$

with:

$$\begin{aligned} \beta_0 &= \frac{1}{(4\pi)^2} \frac{11}{3} \\ \beta_J &= \frac{4}{(4\pi)^2} \end{aligned} \tag{1.2}$$

where g_c is the 't Hooft canonical coupling constant and $\frac{\partial \log Z}{\partial \log \Lambda}$ is computed to all orders in the 't Hooft Wilsonian coupling constant, g_W , by:

$$\frac{\partial \log Z}{\partial \log \Lambda} = \frac{\frac{1}{(4\pi)^2} \frac{10}{3} g_W^2}{1 + c g_W^2} \tag{1.3}$$

with c a scheme dependent arbitrary constant. At the same time, the beta function for the 't Hooft Wilsonian coupling is exactly one loop:

$$\frac{\partial g_W}{\partial \log \Lambda} = -\beta_0 g_W^3 \tag{1.4}$$

Once the result for $\frac{\partial \log Z}{\partial \log \Lambda}$ to the lowest order in the canonical coupling

$$\frac{\partial \log Z}{\partial \log \Lambda} = \frac{1}{(4\pi)^2} \frac{10}{3} g_c^2 + \dots \tag{1.5}$$

is inserted in eq. (1.1), it implies the correct value of the first and second perturbative coefficients of the beta function [2–5]:

$$\begin{aligned} \frac{\partial g_c}{\partial \log \Lambda} &= -\beta_0 g_c^3 + \left(\frac{\beta_J}{4} \frac{1}{(4\pi)^2} \frac{10}{3} - \beta_0 \beta_J \right) g_c^5 + \dots \\ &= -\frac{1}{(4\pi)^2} \frac{11}{3} g_c^3 + \frac{1}{(4\pi)^4} \left(\frac{10}{3} - \frac{44}{3} \right) g_c^5 + \dots \\ &= -\frac{1}{(4\pi)^2} \frac{11}{3} g_c^3 - \frac{1}{(4\pi)^4} \frac{34}{3} g_c^5 + \dots \end{aligned} \tag{1.6}$$

which are known to be universal, i.e. scheme independent.

We present in this paper a more general and simplified computation of the effective action that leads to the exact beta function.

In addition we argue that there is a scheme in which the canonical coupling coincides with a certain definition of the physical effective charge in the inter-quark potential. In this scheme the beta function is given by:

$$\frac{\partial g_{\text{phys}}}{\partial \log r} = \beta_0 \frac{g_{\text{phys}}^3}{1 - \beta_J g_{\text{phys}}^2} \frac{\log(r\Lambda_W)}{\log(r\Lambda_W) - \frac{15}{121}} \tag{1.7}$$

with Λ_W the renormalization group (*RG*) invariant scale in the Wilsonian scheme.

But overall we furnish a mathematical explanation for the exactness of the beta function.

On the mathematical side our result for the beta function is based on localizing on a saddle-point equation for an effective action a version of the large- N loop equation for certain quasi *BPS* Wilson loops, in which the functional integration is, by a change of variables, on the anti-self-dual (*ASD*) part of the curvature of the connection.

The mentioned quasi *BPS* Wilson loops of the large- N pure *YM* theory enjoy non-renormalization properties analogous to the ones of *BPS* Wilson loops of theories with extended supersymmetry [6]. In particular we find that they do not have perimeter and cusp divergences for backtracking cusps.

We may wonder how this localization can be possible. In fact there are examples of non-trivial quantum field theories in which some special observables can be evaluated exactly at a saddle-point. The rationale behind these examples is a version of the localization described in [7], introduced in quantum field theories in [8]. Some of the most interesting cases are the cohomology ring of $d = 2$ *YM* theory [8], the prepotential of $d = 4$ $\mathcal{N} = 2$ *SUSY* gauge theory [9] and more recently some circular Wilson loops in $d = 4$ $\mathcal{N} = 4, 2$ *SUSY* gauge *YM* theories [10]. Localization in quantum field theory is based on cohomological concepts [7, 8] that involve the action of a nilpotent *BRST* differential on equivariantly closed forms. The proof of localization is obtained noticing that we can add freely to an exponentiated equivariantly closed form the *BRST* variation of any globally

defined form without changing its integral. Then the added *BRST* differential is rescaled by a divergent factor to get localization by a semi-classical argument.

In all the mentioned four-dimensional cases the *BRST* charge on which localization is based is the supercharge of a kind of twisted supersymmetry of the action and of the observables. Thus localization, when it applies, applies only to special observables and cannot be extended to all the observables of the theory.

The localization in the large- N pure *YM* theory that we refer to is furnished by a certain version of the large- N loop equation and it is not based at all on a form of twisted *SUSY*, that of course is absent in the *YM* theory.

Rather, and perhaps not surprisingly, the localization of the loop equation on a saddle point for an effective action becomes possible only after exploiting a stringy character of the loop equation for the quasi *BPS* Wilson loops in which the zig-zag symmetry [11] of the Wilson loops plays a key role.

The zig-zag symmetry is the invariance of our quasi *BPS* Wilson loops for the addition of a backtracking arc. By means of the zig-zag symmetry we can add freely to the Wilson loop backtracking arcs of arbitrary length that end into cusps. In a regularized version we can substitute the backtracking arc with the boundary of a tiny strip, of size of the cut-off.

Thus the localization that we describe in the loop equation is a homological concept, rather than cohomological, and its proof is therefore obtained drawing a certain (weighted) arc family on the loop of the loop equation, that at topological level obeys the axioms of topological closed/open string theory [12].

It is of the utmost importance that the zig-zag symmetry be explicitly non-anomalous, i.e. that it holds without fine-tuning of the renormalization scheme, for localization to apply in our version of the loop equation. This is the case for the quasi *BPS* Wilson loops that we refer to, because of the absence of cusp anomaly for backtracking cusps and of perimeter divergence. Since the zig-zag symmetry is the homological equivalent of the cohomological invariance of the cohomology class of a closed form by the addition of the *BRST* differential of a globally defined form, alone it is by far too general to suffice to localize the loop equation, as it is its corresponding notion in cohomology. Indeed we will see that there exist renormalization schemes in which the zig-zag symmetry holds for all the usual unitary Wilson loops, that form a complete set of the theory. It is very unlikely that any form of localization will ever apply to all these observables.¹

Another, more dynamical, crucial ingredient is necessary. In fact, following the analogy with cohomological localization, we need the analog of the property that the action of the theory is an equivariantly closed form. We can rephrase this property by saying that there is a *BRST* symmetry of the action that generates the cohomologically trivial

¹Formally the computations of this paper apply, mutatis mutandis, also to some diagonally embedded Wilson loops whose curvature is, following our conventions, of *SD* type. The associated connection is, in the notation of section2, $B^{\text{phys}} = (A_z + D_u)dz + (A_{\bar{z}} - D_{\bar{u}})d\bar{z}$ and thus it is hermitean. These Wilson loops are physical in the sense that they have perimeter divergence and cusp anomaly and they imply a non-trivial inter-quark potential. Thus, for them, the zig-zag symmetry holds only by fine-tuning of the renormalization scheme. In addition, for them, the Eguchi-Kawai reduction does not lead to the parabolic Higgs bundles of Hitchin type of section3 .

deformations. In our homological language there is no such symmetry of the action, but there is a symmetry of the RG flow of the Wilsonian renormalized effective action. In fact for the quasi BPS Wilson loops the two-dimensional local conformal transformation that generates the holonomically trivial deformation of the loop (see figure(1) and figure(2) of [13]) can be lifted to a local conformal transformation of the four-dimensional theory, because of the quasi BPS constraint (see section2 and section3). The essential reason for this lifting is that a quasi BPS Wilson loop lives on a two-dimensional Riemann surface Σ that is diagonally embedded in the four-dimensional theory.

This conformal transformation moves the Wilsonian renormalized action along the RG trajectory since it is equivalent to add to it, as a local counter-term, the conformal anomaly. The conformal anomaly is computed exactly a posteriori from the exact beta function of the localized effective action.

From this point of view the reason a posteriori for homological localization is that the Wilsonian renormalized effective action flows to the ultraviolet by the local conformal map and thus, being asymptotically free (AF), to vanishing coupling. Then a semi-classical argument implies localization as in the cohomological case.

However we can find a purely homological reason for localization to apply in the loop equation in the ASD variables. As we have anticipated, it turns out that the loop equation can be localized on a critical equation only on a weighted graph [12–14] whose spine, obtained collapsing all the tiny strips to strings, is a Mandelstam graph [15] on which backtracking strings that end into cusps are drawn. On such a graph each string meets transversely the Wilson loop. In addition each string starts and ends into a cusp in such a way that the Wilson loop separates the cusps pairwise into two regions. The effective action on the weighted graph has a stringy character in the sense that its local part is in fact bi-local, since it is localized on pairs of cusps having a string in common. The homological reason for which the loop equation in the ASD variables localizes on the weighted graph is that the usual obstruction to localization in the loop equation, that is the contact term, vanishes at the backtracking cusps of the Mandelstam graph, because of the zig-zag symmetry in our (regularized) version of the loop equation and because of the absence of cusp anomaly for the quasi BPS Wilson loop.

The cusps carry the local degrees of freedom of the theory and correspond to the parabolic points of a dense embedding in the functional integral of twisted parabolic Higgs bundles (of a non-commutative version of the large- N YM theory). From a stringy point of view the cusps are the D -branes at which open strings can end [12]. This also explains a posteriori why, to get localization in the loop equation, a change of variables is needed in which the functional integral is defined at the points rather than at the links of a lattice. This means, in continuum language, that in our version of the loop equation we change variables from the connection to (the ASD part of) the curvature.

Parabolic Higgs bundles have been introduced in the functional integral of pure YM theory in [16] and more recently in the functional integral of $\mathcal{N} = 4$ $SUSY$ YM theory in [17] in their study of the ramified Langlands conjecture. We find that the exact beta function arises localizing the quasi BPS Wilson loops on the moduli of twisted parabolic Higgs bundles of a special type that correspond to the Z_N non-abelian vortices of a partial non-commutative large- N Eguchi-Kawai (EK) reduction [18–22].

The plan of the paper is as follows.

In section2 we show the existence of the quasi *BPS* Wilson loop in large- N pure YM theory starting by analogy with the case of extended supersymmetry.

In section3 we recall the loop equation of the pure YM theory in the *ASD* variables following [1].

In section4 we compute the beta function of the pure large- N YM theory generalizing and simplifying in several ways the computation already presented in [1].

In section5 we exploit the scheme dependence of the beta function of the canonical coupling constant to create a link with the beta function of a certain definition of the physical effective charge [23] in the inter-quark potential in the large- N limit. We also compare our large- N beta function with the numerical results found for $SU(3)$ [23].

In section6 we recall our conclusions.

2 Quasi *BPS* Wilson loops

We recall in this section the properties of (locally) *BPS* Wilson loops [24] originally introduced in the study of $\mathcal{N} = 4$ *SUSY* YM/AdS string duality in [25]. We use the notation and some of the arguments of [6] about non-renormalization properties of locally *BPS* Wilson loops (see figure(1) of [6] for a pictorial representation of perimeter and cuspidal renormalizations of Wilson loops).

It has been argued in [6] that a locally *BPS* Wilson loop in the four-dimensional $\mathcal{N} = 4$ *SUSY* gauge theory:

$$Tr\Psi(BPS) = TrP \exp i \int_C (A_a dx_a(s) + i\phi_b dy_b(s)) \tag{2.1}$$

has no perimeter divergence to all orders in perturbation theory because of the local *BPS* constraint:

$$\sum_a \dot{x}_a^2(s) - \sum_b \dot{y}_b^2(s) = 0 \tag{2.2}$$

Indeed at lowest order of perturbation theory this constraint assures the cancellation of the contribution to the perimeter divergence of the gauge propagator versus the scalar propagator, because of the factor of i^2 in front of the scalar propagator at that order. As far as the perimeter divergence is concerned it is argued in [6] that this cancellation occurs to all orders in perturbation theory, when the locally *BPS* Wilson loop is seen as the dimensional reduction to four dimensions of the ten-dimensional Wilson loop of the ten-dimensional $\mathcal{N} = 1$ *SUSY* YM theory from which the four-dimensional $\mathcal{N} = 4$ *SUSY* theory is obtained.

We report here only the ten-dimensional version of the argument in [6], in which *SUSY* plays no role. In fact the argument is based only on $O(10)$ rotational symmetry, as we show momentarily. In ten dimensions the coefficient of the perimeter divergence of an ordinary unitary Wilson loop at any order in perturbation theory must necessarily contain as a factor a polynomial in the $O(10)$ invariant quantity $\sum_\alpha \dot{x}_\alpha^2(s)$ without a constant term. Indeed

at any order in perturbation theory a generic contribution to the Wilson loop contains a correlator of gauge fields, i.e. a Green function, with tensor indices contracted with a product of monomials in $\dot{x}_\alpha(s)$ at generic insertion points on the loop, labeled by s .

The perimeter divergence arises when all insertion points coincide in such a way that all the arguments of the Green function vanish. In this case the Green function provides a factor that by $O(10)$ rotational invariance must be a polynomial in ten-dimensional Kronecker delta, since all the difference vectors are zero at coinciding points and thus no other tensorial structure can be produced.

This combines with the factors of $\dot{x}_\alpha(s)$ to produce an invariant polynomial in $\sum_\alpha \dot{x}_\alpha^2(s)$ with no constant term since the lowest order contribution is zero by direct computation. But $\sum_\alpha \dot{x}_\alpha^2(s)$ is zero for a *BPS* Wilson loop because of the *BPS* constraint. A naive application of this argument to the four-dimensional *BPS* Wilson loop would imply the absence of the perimeter divergence for this loop on the basis of the $O(10)$ rotational invariance of the theory before the dimensional reduction. Independently of subtleties eventually needed to apply the argument to the dimensionally reduced $d = 4$ Wilson loop in the $\mathcal{N} = 4$ theory we show now that in $\mathcal{N} = 1$ and $\mathcal{N} = 0$ theories Wilson loops exist for which no perimeter divergence occurs because of the $O(4)$ rotational symmetry in the large- N limit. We will show momentarily why in absence of *SUSY* the large- N limit is needed.

The connection that we look for is the following one:

$$B = A + D = (A_z + D_u)dz + (A_{\bar{z}} + D_{\bar{u}})d\bar{z} \tag{2.3}$$

where z, \bar{z}, u, \bar{u} are the four-dimensional complex coordinates and $D_u = \partial_u + iA_u$ is the u component of the covariant derivative. From a two-dimensional point of view B is a non-hermitean connection defined as the sum of a hermitean two-dimensional connection and a Higgs field. Indeed covariant derivatives transform as a Higgs field for gauge transformations. Let us notice also that in the $\mathcal{N} = 1$ theory B gives rise to a locally *BPS* Wilson loop because of the constraint that is implicit in its definition:

$$\begin{aligned} dz(s) &= du(s) \\ d\bar{z}(s) &= d\bar{u}(s) \end{aligned} \tag{2.4}$$

The factor of i in front of the connection in the covariant derivative plays the same role as the factor of i in front of the Higgs field in locally *BPS* Wilson loops of theories with extended *SUSY*. If the derivative term were absent in the covariant derivatives we could invoke the $O(4)$ symmetry of the theory and the local *BPS* constraint to imply the absence of the perimeter divergence as in the $O(10)$ *SUSY* case. Yet, there are derivatives.

In fact if we make the space non-commutative in the u, \bar{u} directions, we can get rid of the derivatives by a gauge transformation because the translations can be absorbed into the gauge transformations in a non-commutative theory [26–28] and more generally in the large- N limit [29].

However non-commutativity breaks the $O(4)$ symmetry in such a way that the property that we are looking for is lost.

Yet, since the limit of infinite non-commutativity is equivalent to the ordinary large- N limit [27, 28], as can be seen for example from the loop equation of the non-commutative theory, and in the ordinary commutative large- N limit certainly $O(4)$ symmetry holds, then in the limit of infinite non-commutativity our Wilson loop has no perimeter divergence.

This holds in the pure YM theory and in the $\mathcal{N} = 1$ YM theory without using $SUSY$. We now show, by an argument based on the Makeenko-Migdal (MM) loop equation [30, 31], that the perimeter divergence and the cusp anomaly for backtracking cusps have to mix for a zig-zag invariant Wilson loop in such a way that the cusp anomaly cancels the part of the perimeter divergence due to the backtracking arcs. An argument of this kind has been anticipated in [6] in their discussion of the zig-zag symmetry on the stringy side of the correspondence AdS/CFT .

A byproduct of this argument is that a zig-zag invariant loop that has no perimeter divergence has no cusp divergence for backtracking cusps, because the zig-zag symmetry requires cancellation between the two contributions and since one of them vanishes the other one has to vanish too.

Introducing the MM loop equation at this stage will allow us to prepare our arguments on charge renormalization too. We can write the MM loop equation for unitary Wilson loops in the large- N YM theory as:

$$\int_{C(x,x)} dx_\alpha \tau \left(\frac{1}{2g^2} \frac{\delta S}{\delta A_\alpha(x)} \Psi(x, x; A) \right) = i \int_{C(x,x)} dx_\alpha \int_{C(x,x)} dy_\alpha \delta^{(4)}(x-y) \tau(\Psi(x, y; A)) \tau(\Psi(y, x; A)) \quad (2.5)$$

where the normalized trace, τ , is the combination of the normalized v.e.v. with the normalized colour trace in the fundamental representation (see for example [1]) and

$$\Psi(x, y; A) = P \exp i \int_{C(x,y)} A_\alpha dx_\alpha \quad (2.6)$$

In the case of loops without self-intersections but with cusps the MM loop equation reduces to:

$$\int_{C(x,x)} dx_\alpha \tau \left(\frac{1}{2g^2} \frac{\delta S}{\delta A_\alpha(x)} \Psi(x, x; A) \right) = i \int_{C(x,x)} dx_\alpha \int_{C(x,x)} dy_\alpha \delta^{(4)}(x-y) \tau(\Psi(x, x; A)) \quad (2.7)$$

Performing the two contour integrations along the loop in the right hand side, we get:

$$\int_{C(x,x)} dx_\alpha \tau \left(\frac{1}{2g^2} \frac{\delta S}{\delta A_\alpha(x)} \Psi(x, x; A) \right) = i \left(La^{-3} + \sum_{\text{cusp}} \frac{\cos \Omega_{\text{cusp}}}{\sin \Omega_{\text{cusp}}} a^{-2} \right) \tau(\Psi(x, x; A)) \quad (2.8)$$

where L is the perimeter of the loop and Ω_{cusp} the cusp angle at a cusp. The perimeter divergence arises by the double integration of the four-dimensional delta function, i.e. of the

contact term, along the loop. However integrating the contact term in a neighborhood of each cusp gives rise to a sub-leading quadratic divergence, since around a cusp we get two independent integrations instead of one, due to the two sides of the cusp. The coefficient of the cusp contribution is proportional to the ratio:

$$\frac{\cos \Omega_{\text{cusp}}}{\sin \Omega_{\text{cusp}}} \tag{2.9}$$

The numerator arises from the scalar product, the denominator from the two independent integrations of the two-dimensional delta function. In the limit in which the cusp angle Ω_{cusp} reaches π , i.e. the cusp backtracks, the cusp contribution in the contact term of the ordinary loop equation is negative and divergent.

If the Wilson loop is zig-zag invariant and the theory is regularized in a zig-zag invariant way this divergence must be fine-tuned to cancel part of the perimeter divergence due to the length of the cusps, in such a way that the only remaining perimeter divergence is the one of the loop without backtracking cusps, since these cusps are irrelevant because of the zig-zag symmetry. Thus we conclude that if a loop is zig-zag symmetric and it has no perimeter divergence it cannot have a cusp anomaly either for backtracking cusps. We can write the *MM* loop equation also for a planar quasi *BPS* Wilson loop:

$$\int_{C(x,x)} d\bar{z} \tau \left(\frac{1}{2g^2} \frac{\delta S}{\delta B_z(x)} \Psi(x, x; B) \right) = i\delta^{(2)}(0) \int_{C(x,x)} d\bar{z} \int_{C(x,x)} dz \delta^{(2)}(z-x) \tau(\Psi(x, z; B)) \tau(\Psi(z, x; B)) \tag{2.10}$$

For quasi *BPS* Wilson loops the contact term in the loop equation is the same as for unitary Wilson loops. Cancellations occur only in the solution.

Hence the *MM* loop equation cannot localize, not even for a quasi *BPS* Wilson loop, because it cannot be regularized in a way that does implement the zig-zag symmetry explicitly. In the next section we write a new loop equation for quasi *BPS* Wilson loops in which the zig-zag symmetry can be implemented explicitly.

3 Localization by homology of the loop equation in the *ASD* variables

Cohomological localization in quantum field theory [8] is based on the fact that the integral of an equivariantly closed differential form depends only on its cohomology class and not on a particular representative:

$$\int (\omega + Q\alpha) = \int \omega \tag{3.1}$$

Thus we have the freedom to add to the exponential of a closed form ω , the action of our quantum field theory, the *BRST* differential of any globally defined form, with an arbitrary coupling t , without changing the integral:

$$\int \exp(\omega + tQ\alpha) \tag{3.2}$$

as it is seen differentiating with respect to t and using eq. (3.1). In this way the saddle-point method applies exactly to the modified action in the limit $t \rightarrow \infty$. Localization of the functional integral on the critical points of $Q\alpha$ then follows. From this argument it is clear that localization is not a property of all the observables of the theory but of only those that are equivariantly closed as the action is.

We would like to find an analog of cohomological localization for the pure YM theory. Unfortunately YM theory has no (twisted) $SUSY$ and thus it is unlikely that this theory will ever admit a cohomological localization of any kind.

As a possible way out we look at the large- N loop equation, the MM equation. In the MM equation the contact term is the obstruction to localization, since in its absence the MM equation would reduce to the insertion in front of the Wilson loop of the classical equation of motion of the theory, that is the saddle-point equation for the action. Of course it is precisely this obstruction that makes the MM equation non-trivial and interesting.

While solving the MM loop equation uniformly for all the Wilson loops is equivalent to find the master field [32], yet a too difficult problem, by localization of the loop equation we mean a transformation of the loop equation into the new form:

$$\tau \left(\frac{\delta \Gamma_q}{\delta \mu(x)} \Psi(x, x; B) \right) = 0 \tag{3.3}$$

for an effective action, Γ_q , for a Wilson loop of a special kind, B , and a trace, τ , involving some new, yet unknown, integration variable, μ . Thus the transformation that we look for must eliminate the contact term, after the change of variables from the gauge connection to the yet unspecified field μ .

Our basic homological property will be the zig-zag symmetry, i.e. the freedom to add to a Wilson loop a backtracking arc without changing its holonomy [11]. For the quasi BPS Wilson loops introduced in the previous section there is no perimeter divergence and no cusp anomaly for backtracking cusps. Thus the basic homological identity holds for them without worrying about regularization and renormalization.

The next (dynamical) ingredient, needed to get homological localization, is a symmetry of the theory that generates the holonomically trivial deformation of a quasi BPS Wilson loop. This is analogous in cohomology to a symmetry of the action that generates a deformation by a co-boundary, i.e. to the action being a closed form. What we need is a symmetry of the RG flow of the (Wilsonian) renormalized effective action. This symmetry arises as follows. There is an action of the two-dimensional conformal group that adds to the quasi BPS Wilson loop the boundary of a tiny strip (see figure(1) and figure(2) of [13]), that lifts to a four-dimensional conformal rescaling because of the diagonal embedding of the planar quasi BPS Wilson loop in the four-dimensional theory.

At this point we can get localization in the loop equation simply by drawing a picture. We must add to the Wilson loop a family of weighted arcs [12] in such a way that all the marked points of the loop are mapped into backtracking cusps (see figure(4) I of [14]). This modification amounts to a (singular) change of the conformal structure around the marked points (see figure(1) and figure(2) of [13]). In the four-dimensional YM theory this generates a conformal anomaly in the effective action, that is equivalent to add a

local counter-term. Since the cusps are backtracking their contribution in the right hand side of a new loop equation in new variables will vanish, provided the new loop equation can be regularized in a way compatible with the zig-zag symmetry. Homological localization follows.

We now write the formulae that correspond to our arguments. We can describe the chain of changes of variables and transformations that lead to large- N homological localization as follows. We start with the large- N YM theory defined on $R^2 \times R_\theta^2$ in the limit of infinite non-commutativity θ , that is known to reproduce the ordinary commutative large- N limit:

$$\begin{aligned} Z &= \int \exp \left(-\frac{N}{2g^2} \sum_{\alpha \neq \beta} \int Tr_f (F_{\alpha\beta}^2) d^4x \right) DA \\ &= \int \exp \left(-\frac{N8\pi^2}{g^2} Q - \frac{N}{4g^2} \sum_{\alpha \neq \beta} \int Tr_f (F_{\alpha\beta}^{-2}) d^4x \right) DA \end{aligned} \quad (3.4)$$

In the second line the classical action is conveniently rewritten as the sum of a topological and a purely ASD term. The topological term Q is the second Chern class, given by:

$$Q = \frac{1}{16\pi^2} \sum_{\alpha \neq \beta} \int Tr_f (F_{\alpha\beta} \tilde{F}_{\alpha\beta}) d^4x \quad (3.5)$$

while the ASD curvature $F_{\alpha\beta}^-$ is defined by:

$$\begin{aligned} F_{\alpha\beta}^- &= F_{\alpha\beta} - \tilde{F}_{\alpha\beta} \\ \tilde{F}_{\alpha\beta} &= \frac{1}{2} \epsilon_{\alpha\beta\gamma\delta} F_{\alpha\beta} \end{aligned} \quad (3.6)$$

Introducing the projectors, P^- and P^+ , the curvature can be decomposed into its ASD and SD components:

$$\begin{aligned} F_{\alpha\beta} &= P^- F_{\alpha\beta} + P^+ F_{\alpha\beta} \\ &= \frac{1}{2} F_{\alpha\beta}^- + \frac{1}{2} F_{\alpha\beta}^+ \end{aligned} \quad (3.7)$$

Notice that the coefficient of the classical action as a functional of the projected ASD curvature is twice the coefficient of the classical action as a functional of the total curvature. This will be important when we will compute the beta function. The generators in the fundamental representation are normalized as:

$$\begin{aligned} Tr_f (T^a T^b) &= \frac{1}{2} \delta_{ab} \\ \sum_a (T^a)_f^2 &= \frac{N^2 - 1}{2N} 1_f \end{aligned} \quad (3.8)$$

We change variables from the connection to the ASD curvature, introducing in the functional integral the appropriate resolution of identity:

$$1 = \int \delta (F_{\alpha\beta}^- - \mu_{\alpha\beta}^-) D\mu_{\alpha\beta}^- \quad (3.9)$$

The partition function thus becomes:

$$\begin{aligned}
 Z &= \int \exp \left(-\frac{N8\pi^2}{g^2} Q - \frac{N}{4g^2} \sum_{\alpha \neq \beta} \int Tr \left(\mu_{\alpha\beta}^{-2} \right) d^4x \right) \\
 &\quad \times \delta \left(F_{\alpha\beta}^- - \mu_{\alpha\beta}^- \right) D\mu_{\alpha\beta}^- DA
 \end{aligned} \tag{3.10}$$

We can write the partition function in the new form:

$$\begin{aligned}
 Z &= \int \exp \left(-\frac{N8\pi^2}{g^2} Q - \frac{N}{4g^2} \sum_{\alpha \neq \beta} \int Tr \left(\mu_{\alpha\beta}^{-2} \right) d^4x \right) \\
 &\quad \times Det'^{-\frac{1}{2}} \left(-\Delta_A \delta_{\alpha\beta} + D_\alpha D_\beta + iad_{\mu_{\alpha\beta}^-} \right) D\mu_{\alpha\beta}^-
 \end{aligned} \tag{3.11}$$

where the integral over the gauge connection of the delta function has been now explicitly performed:

$$\begin{aligned}
 \int DA_\alpha \delta \left(F_{\alpha\beta}^- - \mu_{\alpha\beta}^- \right) &= |Det'^{-1} (P^- d_A \wedge)| \\
 &= Det'^{-\frac{1}{2}} \left((P^- d_A \wedge)^* (P^- d_A \wedge) \right) \\
 &= Det'^{-\frac{1}{2}} \left(-\Delta_A \delta_{\alpha\beta} + D_\alpha D_\beta + iad_{F_{\alpha\beta}^-} \right)
 \end{aligned} \tag{3.12}$$

and, by an abuse of notation, the connection A in the determinants denotes the solution of the equation $F_{\alpha\beta}^- - \mu_{\alpha\beta}^- = 0$. The $'$ superscript requires projecting away from the determinants the zero modes due to gauge invariance, since gauge fixing is not yet implied, though it may be understood if we like to.

We refer to the determinant in the preceding equation as to the localization determinant, because it arises localizing the gauge connection on a given level, $\mu_{\alpha\beta}^-$, of the ASD curvature. Let us notice the unusual spin term $iad_{F_{\alpha\beta}^-}$ as opposed to the one that arises in the background field method $2iad_{F_{\alpha\beta}}$. In the background field method the quadratic form:

$$\frac{1}{2g^2} \int d^4x Tr (F_{\alpha\beta})^2 \tag{3.13}$$

is expanded around a solution of the equation of motion. This produces the factor of two in the spin term. The localization determinant arises instead from the expansion of the quadratic form:

$$\lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \int d^4x Tr (F_{\alpha\beta}^- - \mu_{\alpha\beta}^-)^2 \tag{3.14}$$

that arises in a definition of the delta function around the background:

$$F_{\alpha\beta}^- = \mu_{\alpha\beta}^- \tag{3.15}$$

The corresponding shift in the curvature explains why the coefficient of the spin term is one half of the one in the background field method. The occurrence of $F_{\alpha\beta}^-$ is due instead to the projector on the ASD part of the curvature.

After this change of variables the functional integral is defined on the plaquettes rather than on the links of a lattice, in a lattice regularization of the theory that will be needed later.

We need another change of variables to a holomorphic gauge, in order to write the loop equation in the *ASD* variables in a way convenient to our further developments. This introduces a holomorphic anomaly in the functional integral as a Jacobian. The choice of a holomorphic gauge is required for the following reason. We want to reduce the *MM* loop equation, that is obviously defined on loops, to a critical equation defined on points. There is a canonical way to associate to a loop a point, via the evaluation of a residue. The change of variables to the holomorphic gauge is meant to produce the Cauchy kernel in the loop equation, that in turn can be evaluated as a regularized residue in a proper regularization. In fact the very idea of localization in the loop equation has a holographic interpretation [1] in which to a loop (with a marked point), on which the one dimensional quantum theory on the bulk lives, it is associated a critical equation at the marked point, on which the zero dimensional localized theory on the boundary lives.

We describe the holomorphic gauge as follows. We can interpret the *ASD* relations:

$$F_{\alpha\beta}^- - \mu_{\alpha\beta}^- = 0 \tag{3.16}$$

as an equation for the curvature of the non-Hermitian connection $B = A + D = (A_z + D_u)dz + (A_{\bar{z}} + D_{\bar{u}})d\bar{z}$ and a harmonic constraint for the Higgs field $\Psi = -iD = -i(D_u dz + D_{\bar{u}} d\bar{z}) = \psi + \bar{\psi}$ [33]:

$$\begin{aligned} F_B - \mu &= 0 \\ \bar{F}_B - \bar{\mu} &= 0 \\ d_A^* \Psi - \nu &= 0 \end{aligned} \tag{3.17}$$

that can also be written as:

$$\begin{aligned} F_B - \mu &= 0 \\ \bar{\partial}_A \psi - n &= 0 \\ \partial_A \bar{\psi} - \bar{n} &= 0 \end{aligned} \tag{3.18}$$

where the fields μ, ν, n are suitable linear combinations of the *ASD* components $\mu_{\alpha\beta}^-$. The resolution of identity in the functional integral then reads:

$$1 = \int \delta(F_B - \mu) \delta(\bar{\partial}_A \psi - n) \delta(\partial_A \bar{\psi} - \bar{n}) D\mu Dn D\bar{n} \tag{3.19}$$

where the measure $D\mu$ is interpreted in the sense of holomorphic matrix models [34], employed in the study of the chiral ring of $\mathcal{N} = 1$ *SUSY* gauge theories [35].

This interpretation of the measure $D\mu$ seems to be needed to get the correct counting, in order to reproduce the perturbative beta function, of (complex) zero modes in the effective action. The holomorphic gauge is defined as the change of variables for the connection B , in which the curvature of B is given by the field μ' , obtained from the equation:

$$F_B - \mu = 0 \tag{3.20}$$

by means of a complexified gauge transformation $G(x; B)$ that puts $B = b + \bar{b}$ in the gauge $\bar{b} = 0$:

$$\bar{\partial}b_z = -i\mu' \quad (3.21)$$

where $\mu' = G\mu G^{-1}$ (the factor of i occurs because eq. (3.21) is written in complex coordinates).

Employing eq. (3.19) as a resolution of identity in the functional integral, the partition function becomes:

$$\begin{aligned} Z = & \int \delta(F_B - \mu) \delta(\bar{\partial}_A \psi - n) \delta(\partial_A \bar{\psi} - \bar{n}) \exp\left(-\frac{N}{2g^2} S_{YM}\right) \\ & \times \frac{D\mu}{D\mu'} D b D \bar{b} D \mu' D n D \bar{n} \end{aligned} \quad (3.22)$$

The integral over b, \bar{b} is the same as the integral over the four A_α . The resulting functional determinants, together with the Jacobian of the change of variables to the holomorphic gauge, are absorbed into the definition of Γ .

Γ plays here the role of a classical action, since it must be still integrated over the fields μ', n, \bar{n} . We call Γ the classical *ASD* action, as opposed to the quantum *ASD* effective action, Γ_q . Γ is given by:

$$\begin{aligned} \Gamma = & \frac{N8\pi^2}{g^2} Q + \frac{N}{g^2} \int Tr_f (F_{01}^{-2} + F_{02}^{-2} + F_{03}^{-2}) d^4x \\ & + \log Det'^{-\frac{1}{2}} \left(-\Delta_A \delta_{\alpha\beta} + D_\alpha D_\beta + iad_{\mu_{\alpha\beta}^-} \right) - \log \frac{D\mu}{D\mu'} \end{aligned} \quad (3.23)$$

with:

$$\begin{aligned} \mu^0 &= F_{01}^- \\ n + \bar{n} &= F_{02}^- \\ i(n - \bar{n}) &= F_{03}^- \end{aligned} \quad (3.24)$$

Although Γ is the classical action in the *ASD* variables it contains already quantum corrections because of the Jacobian of the change of variables. It turns out that its divergent part coincides with the divergent part of the Wilsonian localized quantum effective action, after the inclusion of zero modes. Until now the theory is still four dimensional. Taking functional derivatives with respect to the *ASD* field we get, for a planar quasi *BPS* loop, the loop equation:

$$\begin{aligned} 0 = & \int D\mu' Tr \frac{\delta}{\delta\mu'(w, 0)} (\exp(-\Gamma) \Psi(x, x; b))_{zz} \\ = & \int D\mu' \exp(-\Gamma) \left(Tr \left(\frac{\delta\Gamma}{\delta\mu'(w, 0)} \Psi(x, x; b) \right) \right. \\ & \left. - \int_{C(x,x)} dy_z \frac{1}{2} \delta^{(2)}(0) \bar{\partial}^{-1}(w-y) Tr(\lambda^a \Psi(x, y; b) \lambda^a \Psi(y, x; b)) \right) \end{aligned}$$

$$\begin{aligned}
&= \int D\mu' \exp(-\Gamma) \left(\text{Tr} \left(\frac{\delta\Gamma}{\delta\mu'(w,0)} \Psi(x,x;b) \right) \right. \\
&\quad - \int_{C(x,x)} dy_z \frac{1}{2} \delta^{(2)}(0) \bar{\partial}^{-1}(w-y) \left(\text{Tr}(\Psi(x,y;b)) \text{Tr}(\Psi(y,x;b)) \right. \\
&\quad \left. \left. - \frac{1}{N} \text{Tr}(\Psi(x,y;b) \Psi(y,x;b)) \right) \right) \tag{3.25}
\end{aligned}$$

where in our notation we have omitted the integrations $DnD\bar{n}$ since they are irrelevant in the loop equation because the curvature of B depends only on μ . In the large- N limit it reduces to:

$$\begin{aligned}
\tau \left(\frac{\delta\Gamma}{\delta\mu'(w,0)} \Psi(x,x;b) \right) = \\
\int_{C(x,x)} dy_z \frac{1}{2} \delta^{(2)}(0) \bar{\partial}^{-1}(w-y) \tau(\Psi(x,y;b)) \tau(\Psi(y,x;b)) \tag{3.26}
\end{aligned}$$

The quadratically divergent factor $\delta^{(2)}(0)$ arises as follows. The functional derivative at a point produces a factor of $\delta^{(4)} = \delta^{(2)} \times \delta^{(2)}$ at that point because the fields are four dimensional. One factor of $\delta^{(2)}$ is convoluted with the kernel of $\bar{\partial}^{-1}$, while the other factor produces $\delta^{(2)}(0)$ since the loop is assumed to lie on a plane.

Now it is natural to perform a partial *EK* reduction from four to two dimensions. Let us describe what in fact the partial *EK* reduction means in this context. We already observed that we can absorb the translations into a gauge transformation in the four-dimensional non-commutative theory along the two non-commutative directions [26–28], that are the one transverse to the plane over which the loop lies. As a result the classical action looks two dimensional, in the sense that the space-time dependence of the fields is two dimensional, despite the fact that the theory is truly four dimensional. The four-dimensional information is hidden in the central extension $H = \frac{1}{\theta}$ that shows up in the curvature of the connection because of the non-commutativity of the partial derivatives in the non-commutative directions [26–28]. However the loop equation, being gauge invariant, is unchanged by this special gauge choice if the gauge is chosen after taking the functional derivatives. This means that the quadratically divergent factor $\delta^{(2)}(0)$ occurs in the four-dimensional loop equation even in a gauge in which the fields are constant in the non-commutative directions.

If however we were to write the loop equation for the theory already reduced to two dimensions we would miss the $\delta^{(2)}(0)$ divergent factor in the right hand side, because we would get just one factor of $\delta^{(2)}$ by taking functional derivatives, instead of the factor of $\delta^{(4)}$.

Now we use the fact that in the non-commutative theory the integral over the non-commutative directions can be represented as a (colour) trace [26]:

$$\int d^2x_T = \frac{2\pi}{H} \text{Tr} \tag{3.27}$$

Hence the non-commutative classical action of the reduced theory gets a volume factor of $V_2 = 2\pi\theta = \frac{2\pi}{H}$ because of the gauge choice. The equation of motion of the reduced theory

is therefore multiplied by this volume factor. We can divide both sides of the loop equation by this volume factor in the reduced theory in such a way that the equation of motion is normalized as in the four-dimensional theory. Then the inverse volume will appear in the right hand side instead of the factor $\delta^{(2)}(0)$. We can compensate this fact by rescaling the classical action by a factor of N_2^{-1} [29], with $N_2 = V_2 \delta^{(2)}(0)$, in such a way that the factor of $\frac{V_2}{N_2}$ in the reduced classical action produces the factor of $\delta^{(2)}(0) = \frac{N_2}{V_2}$ once carried to the right hand side of the loop equation. Of course in all this discussion we are implicitly assuming that the trace of the reduced theory includes now the non-commutative degrees of freedom.

Thus the reduced classical action of the twisted *EK* reduced theory in the *ASD* variables is given by:

$$\Gamma = \frac{N8\pi^2}{N_2g^2}Q + \frac{N}{g^2} \frac{2\pi}{N_2H} \int Tr_f (F_{01}^{-2} + F_{02}^{-2} + F_{03}^{-2}) d^2x + \log Det'^{-\frac{1}{2}} \left(-\Delta_A \delta_{\alpha\beta} + D_\alpha D_\beta + iad_{\mu\alpha\beta} \right) - \log \frac{D\mu}{D\mu'} \quad (3.28)$$

where the trace in the functional determinants has to be interpreted coherently with the partial *EK* reduction.

The reduced loop equation is then:

$$\tau \left(\frac{\delta\Gamma}{\delta\mu'(w)} \Psi(x, x; b) \right) = \int_{C(x,x)} dy_z \frac{1}{2} \bar{\partial}^{-1}(w - y) \tau(\Psi(x, y; b)) \tau(\Psi(y, x; b)) \quad (3.29)$$

It is interesting to observe that the reducing *EK* factor cancels exactly the quadratic divergence obtained evaluating the action on parabolic Higgs bundles, once it is assumed that the transverse and longitudinal cut-off are equal. But this is necessary to keep our $O(4)$ symmetry. The partial *EK* reduction is not strictly needed for our arguments and for computing the beta function, but it is a convenient technical tool. It allows us to avoid overall infinite factors to appear in our formulae and by this very reason explains how apparently singular objects as the parabolic Higgs bundles survive in the large- N limit of the functional integral. In this respect we should mention a different point of view between our treatment of parabolic Higgs bundles and the one in [17]. In fact in the mathematical literature we can think of parabolic bundles in two slightly different ways. We can think that they are defined on a compact surface with a divisor and a parabolic structure that belongs to the surface. This is our point of view. Or we can think that they arise as boundary conditions on a surface with boundary. This is the point of view in [17].

After the *EK* reduction from four to two dimensions we perform a conformal compactification in such a way that the theory now is defined over a two-dimensional sphere. In the four-dimensional Euclidean theory this amounts to a compactification from R^4 to $S^2 \times S^2$. In the four-dimensional theory in ultra-hyperbolic signature the conformal compactification is instead from the Minkowski space-time M to $\frac{S^2 \times S^2}{Z_2}$ [36]. This adds to the local part of the effective action at most a finite conformal anomaly, that can be ignored.

It is clear that the contour integration in the quantum term of the loop equation includes the pole of the Cauchy kernel. We need therefore a gauge invariant regularization. The natural choice consists in analytically continuing the loop equation from Euclidean to Minkowskian space-time. Thus $z \rightarrow i(x_+ + i\epsilon)$. It is at the heart of the modern Euclidean approach to quantum field theory that this analytic continuation be in fact possible. This regularization has the great virtue of being manifestly gauge invariant. In addition this regularization is not loop dependent.

The result of the $i\epsilon$ regularization of the Cauchy kernel is the sum of two distributions, the principal part plus a one-dimensional delta function (for simplicity we omit the underscript of x_+ in the following):

$$\frac{1}{2}\bar{\partial}^{-1}(w_x - y_x + i\epsilon) = (2\pi)^{-1} (P(w_x - y_x)^{-1} - i\pi\delta(w_x - y_x)) \quad (3.30)$$

The loop equation thus regularized looks like:

$$\tau\left(\frac{\delta\Gamma}{\delta\mu'(w)}\Psi(x, x; b)\right) = \int_{C(x,x)} dy_x (2\pi)^{-1} (P(w_x - y_x)^{-1} - i\pi\delta(w_x - y_x)) \times \tau(\Psi(x, y; b))\tau(\Psi(y, x; b)) \quad (3.31)$$

The right hand side of the loop equation contains now two contributions. A delta-like one dimensional contact term, that is supported on closed loops and a principal part distribution that is supported on open loops. Since by gauge invariance it is consistent to assume that the expectation value of open loops vanishes, the principal part does not contribute and the loop equation reduces to:

$$\tau\left(\frac{\delta\Gamma}{\delta\mu'(w)}\Psi(x, x; b)\right) = \int_{C(x,x)} dy_x \frac{i}{2}\delta(w_x - y_x)\tau(\Psi(x, y; b))\tau(\Psi(y, x; b)) \quad (3.32)$$

Taking $w = x$ and using the transformation properties of the holonomy of b and of $\mu(x)'$, the preceding equation can be rewritten in terms of the connection, B , and the curvature, μ :

$$\tau\left(\frac{\delta\Gamma}{\delta\mu(x)}\Psi(x, x; B)\right) = \int_{C(x,x)} dy_x \frac{i}{2}\delta(x_x - y_x)\tau(\Psi(x, y; B))\tau(\Psi(y, x; B)) \quad (3.33)$$

where we have used the condition that the trace of open loops vanishes to substitute the b holonomy with the B holonomy.

Our argument about the vanishing of the contribution of the principal part becomes tricky in the non-exactly gauge invariant regularization of the loop equation that was considered in [6]. Indeed that regularization allows the contribution of quasi-closed loops, for which of course the principal part contributes too. Yet we will see that our final argument about reducing the loop equation to a saddle-point on the weighted graph is not

affected, not even if the contribution of the principal part is allowed using the regularization in [6]. Indeed the zig-zag symmetry works for both the contact term and the principal part.

We need a lattice version of the continuum loop equation to implement our localization argument. Thus we write the loop equation in the *ASD* variables on a lattice in the partially *EK* reduced theory. On a dense set in the functional integral (in the sense of distributions), the equations:

$$\begin{aligned}
 F_{z\bar{z}} + i[D_u, D_{\bar{u}}] &= i \left(\sum_p \mu_p^0 \delta^{(2)}(x - x_p) - H1 \right) \\
 \bar{\partial}_A(D_u) &= i \sum_p n_p \delta^{(2)}(x - x_p) \\
 \partial_A(D_{\bar{u}}) &= i \sum_p \bar{n}_p \delta^{(2)}(x - x_p)
 \end{aligned} \tag{3.34}$$

define an infinite-dimensional twisted local system or, what is the same, a twisted parabolic Higgs bundle [33, 37–41] on a sphere. The curvature equation involves a central term, $H = \frac{1}{\theta}$, since $i[D_u, D_{\bar{u}}] = -F_{u\bar{u}} - iH1$ in the non-commutative theory. Correspondingly, since the gauge connection has to vanish at infinity, the right hand side has been shifted by $H1$, deforming the resolution of identity of the *ASD* part of the curvature in the functional integral. In the case $n = \bar{n} = 0$, that will be the most relevant for us, we may interpret the preceding equations as vortices equations. In the partially *EK* reduced theory the vortices live at the lattice points where the *ASD* curvature is singular. This means that in the original four-dimensional theory they form two-dimensional vortices sheets [16, 17]. The loop equation on our lattice now reads:

$$\begin{aligned}
 \tau \left(\frac{\delta\Gamma}{\delta\mu(x_p)} \Psi(x_p, x_p; B) \right) &= \\
 \int_{C(x_p, x_p)} dy_z \frac{1}{2} \bar{\partial}^{-1}(x_p - y) \tau(\Psi(x_p, y; B)) \tau(\Psi(y, x_p; B)) & \tag{3.35}
 \end{aligned}$$

and correspondingly for the analytic continuation to Minkowskian space-time:

$$\begin{aligned}
 \tau \left(\frac{\delta\Gamma}{\delta\mu'(w_{x_q})} \Psi(x_q, x_q; B) \right) &= \\
 \int_{C(x_q, x_q)} dy_x (2\pi)^{-1} (P(w_{x_q} - y_x)^{-1} - i\pi\delta(w_{x_q} - y_x)) & \\
 \times \tau(\Psi(x_q, y; B)) \tau(\Psi(y, x_q; B)) & \tag{3.36}
 \end{aligned}$$

Notice that a smooth marked point gives a non-trivial contribution to the right hand side of the loop equation in the *ASD* variables either in the continuum or on the lattice, that is finite for the contact term and logarithmically divergent for the principal part (in the regularization in [6]). However around a backtracking cusp the contributions of the two sides of the asymptotes to the cusp cancel each other for the contact term:

$$\int_{C(x_q, x_q)} dy_x(s) \delta(w_{x_q}(s_{\text{cusp}}) - y_x(s)) = \frac{1}{2} \left(\frac{\dot{w}_{x_q}(s_{\text{cusp}}^+)}{|\dot{w}_{x_q}(s_{\text{cusp}}^+)|} + \frac{\dot{w}_{x_q}(s_{\text{cusp}}^-)}{|\dot{w}_{x_q}(s_{\text{cusp}}^-)|} \right) \tag{3.37}$$

because of the opposite sign of $\dot{w}_{x_q}(s_{\text{cusp}}^+)$ and $\dot{w}_{x_q}(s_{\text{cusp}}^-)$ on the two sides of the backtracking cusp. For the principal part the same argument applies because of the opposite orientations of the asymptotes and because both the cusp asymptotes are approached either from below or from above:

$$\left| \int_{C(x_q, x_q)} dy_x(s) P(w_{x_q}(s_{\text{cusp}}) - y_x(s))^{-1} \right| = \frac{1}{2} \left| \int ds^+ \frac{\dot{y}_x(s^+)}{|w_{x_q}(s_{\text{cusp}}) - y_x(s^+)|} + \int ds^- \frac{\dot{y}_x(s^-)}{|w_{x_q}(s_{\text{cusp}}) - y_x(s^-)|} \right| \quad (3.38)$$

Thus if every marked point of the loop can be transformed into a backtracking cusp we can complete our argument about localization. But this is precisely the effect of our lattice, since marked points of the loop contribute to the loop equation in the lattice theory only if they coincide with the lattice points. Thus we can simply draw our backtracking strings from the loop to the lattice points in order to transform all the marked points into cusps.

Hence we may say that open strings solve the YM loop equation for the quasi BPS Wilson loops, in the sense that they localize the loop equation on a saddle point for an effective action.

We now show that the cusps must be paired by the backtracking strings. Independently on the string gluing axioms [12] we can understand directly from the gluing properties of the functional integral why it must be so.

For a pictorial representation of the construction that we describe see [14]. We can represent the partition function on a sphere by gluing the two cups with the Wilson loop and the marked point on the loop in common. To apply our vanishing result for the contact term to the loop equation on each cup we must connect the same marked point on each cup with a backtracking string ending into a cusp. If we now glue the two cups, the cusps come in pairs on the glued surface and are linked by a string transverse to the loop that intersects the loop at the marked point. These are precisely the string gluing axioms [12] for arc families at topological level (see figure4 I of [14]). Thus we come to the conclusion that if we combine the gluing properties of the functional integral with the vanishing requirement for the contact term, i.e. the requirement of localization, we get the string gluing axioms at topological level. A subtle point arises as follows. When we add a backtracking arc to a marked point we create a loop self-intersection that may contribute extra terms to the loop equation. However the self-intersection may be regularized adding instead tiny strips having a common asymptote to the two cusps. In this case the loop remains simple and the cusps are still pairwise identified (see figure4 III of [14]). Thus the proper graph to get localization is a weighted graph [12] whose spine is a Mandelstam graph (see figure4 I of [14]). The weights in our language are the sizes of the strips.

Two more comments are in order. Our change of variables from the connection to the ASD part of the curvature in the light-cone gauge [42] in the $\mathcal{N} = 1$ YM theory is the Nicolai map [43, 44].²

²We thank Gabriele Veneziano for pointing out this feature to us. Indeed it turns out that in the $\mathcal{N} = 1$ case our loop equation leads to localization on the non-commutative instantons in the limit of infinite

It is one of the cornerstones of the Euclidean field theory that the analytic continuation to Minkowskian space-time can always be performed. In fact if we want to keep the *ASD* structure real we must continue to ultra-hyperbolic signature. If we perform the compactification to $S^2 \times S^2$ in Euclidean signature we get the double cover of the conformal compactification in ultra-hyperbolic signature [36]. Thus if we require that the Euclidean equation of *ASD* type be continued to ultra-hyperbolic signature we must take into account this global constraint.

We are now ready to write the quantum effective action for our localized version of the loop equation. A subtle point arises about the cut-off of this effective action. We recall that the introduction of a lattice is essential for localization, since it allows us to transform every non-trivial marked lattice point into a backtracking cusp. Now, because all lattice points are pairwise linked by strings and the strings are transverse to the loop, there are the same number of lattice points inside and outside the loop.

Hence we need different cut-off scales to fix the two different areas of the cups in which the loop divides the sphere. Thus the topological axioms imply different cut-offs at the cusps of the two cups of the sphere in the quantum effective action.

In fact an equal cut-off would lead to overcounting in the normalization of the classical action, that has to be the same as the one for a unique marked point, up to terms vanishing with the ultraviolet cut-off. Hence the local part of the effective action, as a consequence of the stringy nature of localization in the loop equation, is in fact bi-local with two local fields (differing in fact only by a gauge transformation not connected to the identity since they are originally associated to the same marked point by gluing) living at different scales, one at the ultraviolet cut-off and one at the infrared cut-off. The field at the ultraviolet, but not the one at the infrared, affects the renormalization of the Wilsonian coupling constant, as it is expected from its very definition. Instead the field at the infrared together with the one at the ultraviolet affects the renormalization of the canonical coupling constant.

Finally we draw our weighted graph [12, 14, 15], that is made by two charts with the boundary loop in common. The charts are a conformal transformation of two topological disks with marked points on the boundary (see figure(1) and figure(2) of [13]). This introduces a conformal transformation in the *EK* two-dimensional reduced theory. However this transformation is four dimensional in the original theory because of the *BPS* constraint extended to a neighborhood of the marked points:

$$\begin{aligned} dz &= du \\ d\bar{z} &= d\bar{u} \end{aligned} \tag{3.39}$$

Thus the four-dimensional metric changes conformally and the effective action changes by the appropriate conformal anomaly. This means that, in addition to the explicit cut-off dependence, the effective action on the weighted graph is related to the one on the sphere with marked points by the addition of a divergent conformal anomaly, because of the singularity of the conformal transformation. This divergent conformal anomaly plays a key

non-commutativity and to the already known exact Novikov-Shifman-Vainshtein-Zakharov (*NSVZ*) [45] beta function in the large- N limit [46].

role in our computation of the contribution of the anomalous dimension in the canonical beta function and more generally in the interpretation a posteriori of localization as a RG flow to the ultraviolet.

The loop equation on the weighted graph reduces to the localized form:

$$0 = \tau \left(\frac{\delta \Gamma_q}{\delta \mu(x_p)} \Psi(x_p, x_p; B) \right) \tag{3.40}$$

which we refer to as the master equation.

4 Effective action and exact beta function

In this section we compute the local divergent part of the quantum effective action for the purpose of obtaining the beta function. While the computation and the results are already essentially contained in [1], the calculations here are considerably simplified and several unnecessary constraints are removed.

In particular we compute the beta function in the sector of the YM theory in which the second parabolic Chern class, Q , is negligible with respect to the quadratic divergence induced by a non-trivial parabolic divisor in the four-dimensional theory. This is equivalent to require that the reduced topological term, $\frac{Q}{N_2}$, vanishes when $N_2 \rightarrow \infty$, the reduced EK action being finite on the parabolic divisor. In [1] it was assumed instead the stronger constraint that the parabolic Chern class vanishes.

Moreover in [1] it was assumed that the structure of the fibration of parabolic Higgs bundles in the four-dimensional theory implies, as a consequence of the vanishing of the first and second parabolic Chern classes, the value $\frac{2kN(2\pi)^2}{g^2}$ for the reduced EK classical action at leading $\frac{1}{N}$ order, that corresponds to a superposition of k Z_N vortices and k anti-vortices of lowest vortex number. The anti-vortices occur in [1] because a Wilson loop in the adjoint representation was considered there, that factorizes in the large- N limit into the product of Wilson loops in the fundamental and in the conjugate representation.

Also this constraint is removed, since no four-dimensional structure of the parabolic fibration survives the EK reduction, because of the vanishing of $\frac{Q}{N_2}$. Yet, in this section, the correct normalization of the EK classical action, that reproduces the universal coefficients of the perturbative the beta function, is obtained automatically for vortices of any vortex number. Finally we find that our construction works directly for a quasi BPS Wilson loop in the fundamental representation without introducing the adjoint representation, unless we wish to do so.

The quantum effective action for a quasi BPS Wilson loop in the fundamental representation has the following structure:

$$\exp(-\Gamma_q) = \int d(\text{zero - modes}) \exp(-\Gamma + \text{Conformal Anomaly}) \tag{4.1}$$

In the evaluation of the local divergent part of Γ_q the global features of the weighted graph over which Γ_q is defined are irrelevant, since the divergences of Γ , the conformal anomaly and even the zero modes are locally defined, as we will see momentarily. Γ is the classical

effective action in the *ASD* variables, that includes the non-zero modes of the Jacobian of the change of variables. The local part of Γ differs by the quantum effective action on the weighted graph by a conformal anomaly, since the weighted graph is a conformal image of the sphere with the marked divisor over which Γ is defined.

Computationally the conformal anomaly plays an important role later, in the calculation of the beta function for the canonical coupling, but for the moment can be ignored. Γ_q may contain the contribution of zero modes that are not included in Γ . Γ contains a globally defined part that arises by the topological term in the classical *YM* action. This topological term never contributes to the quantum effective action in every sector in which it is finite, since in the reduced theory it is divided by the divergent factor N_2 . This observation makes irrelevant the whole discussion made in [1] of the four-dimensional constraints that have to be satisfied in order to make the parabolic Chern number vanishing in a way compatible with localization on vortices.

There is a precise relation between N_2 and the quadratic divergence at the parabolic points dictated by the requirement that the cut-off in the longitudinal z plane and in the transverse u plane be equal. In fact the *EK* reduction requires the relation $N_2 = \frac{H}{2\pi} \left(\frac{\Lambda_T}{2\pi}\right)^2$ where $\frac{\Lambda_T}{2\pi} = \frac{1}{a_T}$ is the ultraviolet cut-off in the transverse directions of the *EK* partial reduction. But it must be $\Lambda_T = \Lambda_L$ to keep the $O(4)$ symmetry of the large- N theory, that in turn ensures the non-renormalization properties of our quasi *BPS* Wilson loop.

In the reduced *EK* theory Γ is defined on the two-dimensional sphere with marked points, before mapping conformally to the weighted graph. Thus the lattice divisor that is the support of the local part of Γ is not uniform in general, since this lattice has the same number of points in the interior and in the exterior of the Wilson loop. Indeed the weighted graph must have the same number of cusps in the interior and in the exterior of the Wilson loop. As a consequence the lattice cut-off differs for each pair of marked points (cusps on the weighted graph) linked by a backtracking string. One of these points has a cut-off in the ultraviolet and we call the set of all such points the ultraviolet divisor. The other one has a cut-off in the infrared and we call the set of all such points the infrared divisor.

Though Γ is the classical action in the *ASD* variables, Γ is not finite because generically the non-zero modes of the Jacobian determinant of the change of variables plus the gauge-fixing ghost determinant introduce already some divergences. These divergences can be computed exactly, being only one loop. They correct the classical *YM* action in Γ by a Z^{-1} factor that is explicitly displayed below:

$$\begin{aligned}
 & \left(\frac{N}{2g_W^2} - \left(2 - \frac{1}{3}\right) \frac{N}{(4\pi)^2} \log \left(\frac{\Lambda}{\tilde{\Lambda}} \right) \right) \sum_{\alpha \neq \beta} \int d^4x Tr_f \left(2 \left(\frac{1}{2} F_{\alpha\beta}^- \right)^2 \right) \\
 &= \left(\frac{N}{2g_W^2} - \frac{5}{3} \frac{N}{(4\pi)^2} \log \left(\frac{\Lambda}{\tilde{\Lambda}} \right) \right) \sum_{\alpha \neq \beta} \int d^4x Tr_f \left(2 \left(\frac{1}{2} F_{\alpha\beta}^- \right)^2 \right) \\
 &= \frac{N}{2g_W^2} Z^{-1} \sum_{\alpha \neq \beta} \int d^4x Tr_f \left(2 \left(\frac{1}{2} F_{\alpha\beta}^- \right)^2 \right) \tag{4.2}
 \end{aligned}$$

where Z^{-1} is given by:

$$Z^{-1} = 1 - \frac{10}{3} \frac{1}{(4\pi)^2} g_W^2 \log\left(\frac{\Lambda}{\bar{\Lambda}}\right) \tag{4.3}$$

and we have added to g the under-script W to stress that our computation here refers to the Wilsonian coupling constant. This formula for Z is actually exact to all orders in the Wilsonian coupling constant, up to finite terms. The divergent contribution in Γ is due entirely to the non-zero modes of the localization determinant and of the ghost determinant. The contribution of the holomorphic anomaly in Γ vanishes in any gauge in which μ can be triangularized. Indeed we can reach this gauge either in a unitary basis or in a holomorphic basis. The unitary and the holomorphic basis induce the same Vandermonde determinant of the eigenvalues of μ in the functional measure and thus the holomorphic anomaly vanishes in this gauge.

Γ does not reproduce the perturbative one-loop beta function. This is due to the peculiar spin term that occurs in the localization determinant as opposed to the spin term of the gluons determinant in the background field calculation. The Z^{-1} factor is responsible in the pure YM case of the occurrence of an anomalous dimension in the formula for the canonical beta function as opposed to the $\mathcal{N} = 1$ $SUSY YM$ case. The Z^{-1} factor is not present in the $\mathcal{N} = 1$ $SUSY YM$ theory [46] because in that case the existence of the Nicolai map ensures the cancellation of the non-zero modes of the determinant of the change of variables versus the gluinos determinant in the light-cone gauge [43, 44].

Generically there are no normalizable zero modes because the gauge connection and the Higgs field are both singular at the parabolic points. The resulting pairing in the symplectic volume form in function space:

$$\int d^4x Tr(\delta A \wedge \delta A) \tag{4.4}$$

is divergent. Thus generically in function space and in absence of normalizable zero modes a less negative beta function than the correct one is obtained. However there is a special locus of the Higgs field for which normalizable zero modes exist. This locus corresponds to zeros of the Higgs field, i.e. to vortices equations. The following volume form, $\omega \wedge \bar{\omega}$, is then finite, because the singularity of the gauge connection is compensated by the zero of the Higgs field:

$$\omega = \int d^4x Tr(\delta A_z \wedge \delta D_u) = \frac{2\pi}{H} \int d^2x Tr(\delta A_z \wedge \delta D_u) \tag{4.5}$$

Since the Higgs field is smooth, for vortices the Hitchin equations of ASD type [33, 37–41] reduce to the case $n_p = 0$. Correspondingly the eigenvalues of the ASD curvature are quantized in such a way that the local holonomy carries a Z_N charge in the fundamental representation. This quantization follows by the existence of the zeros of the Higgs field. At the same time the Z_N local holonomy fixes the normalization of the eigenvalues of the curvature (up to a shift by large gauge transformations with trivial local holonomy) and thus the value of the action (up to these shifts, see below) without any extra condition.

For a Z_N vortex of charge k in a $SU(N)$ orbit we get $N - k$ eigenvalues of the curvature equal to $\frac{2\pi k}{N}$ and k eigenvalues equal to $\frac{2\pi(k-N)}{N}$. The trace of the eigenvalues of the ASD curvature in the fundamental representation is thus:

$$(N - k) \left(\frac{2\pi k}{N} \right)^2 + k \left(\frac{2\pi(k - N)}{N} \right)^2 = (2\pi)^2 \frac{k(N - k)}{N} \quad (4.6)$$

In addition each Z_N vortex carries a number of zero modes of the localization determinant equal to the dimension of the adjoint orbit $g\lambda g^{-1}$. We do not include zero modes associated to translations of the vortices since their contribution is sub-leading in $\frac{1}{N}$. The complex dimension of an adjoint orbit for a generic parabolic bundle of rank N is given by:

$$dim = \frac{1}{2} \left(N^2 - \sum_i m_i^2 \right) \quad (4.7)$$

where m_i are the multiplicities of the eigenvalues. For vortices this reduces to:

$$dim = \frac{1}{2} (N^2 - k^2 - (N - k)^2) = k(N - k) \quad (4.8)$$

The classical EK reduced action at one point of the ultraviolet divisor on vortices reads:

$$\frac{N}{g_W^2} (2\pi)^2 \frac{k(N - k)}{N} = \frac{8\pi^2}{2g_W^2} k(N - k) \quad (4.9)$$

while at one point of the infrared divisor reads:

$$\frac{a^2}{\tilde{a}^2} \frac{8\pi^2}{2g_W^2} (k(N - k) + N^2 n^2) \quad (4.10)$$

where the shift n represents the contribution to the eigenvalues of the curvature of a central large gauge transformation that does not affect the dimension of the vortex adjoint orbit. Notice that the action at the infrared divisor is suppressed by a power of the ultraviolet cut-off. We recall that the different cut-off scales arise because there are the same number of cusps internal and external to the Wilson loop and thus the different cut-off scales are the only way to measure different internal and external areas. In particular the external area has to go to infinity in the thermodynamic limit. The existence of two scales seems also necessary in non-perturbative definitions of the renormalization procedure [47]. The local action at the ultraviolet on vortices is renormalized by the Z^{-1} factor and now also by the vortices zero modes. Thus we get for the local part of Γ_q at a vortex of charge k :

$$\begin{aligned} \exp(-\Gamma_q(\text{one - vortex})) &= \exp\left(-\frac{8\pi^2}{2g_W^2} Z^{-1} k(N - k)\right) (Ha^2)^{-\frac{k(N-k)}{2}} \\ &\exp\left(-\frac{a^2}{\tilde{a}^2} \frac{8\pi^2}{2g_W^2} (k(N - k) + N^2 n^2)\right) (H\tilde{a}^2)^{-\frac{k(N-k)}{2}} \end{aligned} \quad (4.11)$$

We omit the contribution of the Vandermonde determinant of the eigenvalues because it is finite. In addition we do not include it in the canonical effective action as we did in [1]

since the canonical normalized action in this paper is not defined by the rescaling of the eigenvalues of the *ASD* curvature as in [1].

Notice that we have not included a Z^{-1} factor in the contribution to Γ at the infrared because there it is finite and it can be set equal to 1 by a convenient choice of the subtraction point. The factor of H in front of a^2 arises from the normalization of the symplectic volume form in the non-commutative theory. There is an analogous contribution at the infrared, that is however finite. a^{-1} is the Pauli-Villars regulator of vortices zero modes at the ultraviolet [48]. The exponent of the Pauli-Villars regulator counts the number of complex vortices zero modes rather than the number of real zero modes according to our interpretation of the $D\mu$ integral as a holomorphic integral. We can avoid the holomorphic counting for the vortices moduli space by noticing that after continuation to ultra-hyperbolic signature the vortices equations are defined on the double cover of $\frac{S^2 \times S^2}{Z_2}$ [36] and that we can use this doubling to define a real pairing of the complex moduli at each location of the vortices.

Alternatively we can consider a Wilson loop in the adjoint representation. Then we get a factorized contribution from the fundamental and conjugate representation in the large- N limit. This implies vortices and anti-vortices in the effective action and a real counting of zero modes by pairing the vortices with the anti-vortices, as opposed to the holomorphic counting. Of course the beta function is unchanged because everything gets doubled.

The renormalization of the Wilsonian coupling constant now follows immediately from the effective action at the ultraviolet:

$$\frac{8\pi^2 k(N-k)}{2g_W^2(\tilde{a})} = 8\pi^2 k(N-k) \left(\frac{1}{2g_W^2(a)} - \frac{1}{(4\pi)^2} \left(2 + \frac{5}{3} \right) \log \left(\frac{\tilde{a}}{a} \right) \right) \quad (4.12)$$

We should notice that without doing the *EK* reduction we would obtain the same beta function once it is observed that the action and the number of zero modes would have been multiplied by the common factor of N_2 .

It is interesting to write the effective action before the *EK* reduction since it will be useful in the computation of the beta function for the canonical coupling. The unreduced effective action reads:

$$\begin{aligned} \exp(-\Gamma_q(\text{one} - \text{vortex})) &= \exp - \left(\frac{2\pi}{Ha^2} \frac{8\pi^2}{2g_W^2} Z^{-1} k(N-k) \right) \\ &\exp - \left(\frac{2\pi}{H\tilde{a}^2} \frac{8\pi^2}{2g_W^2} (k(N-k) + N^2 n^2) \right) \\ &\exp \left(\frac{2\pi}{Ha_T^2} \frac{k(N-k)}{2} \log \left(\frac{1}{Ha^2} \right) \right) \\ &\exp \left(\frac{2\pi}{H\tilde{a}_T^2} \frac{k(N-k)}{2} \log \left(\frac{1}{H\tilde{a}^2} \right) \right) \end{aligned} \quad (4.13)$$

where the factor of $\frac{2\pi}{Ha_T^2}$ is the transverse measure over the zero modes two-dimensional sheet. Indeed, since in the *EK* partially reduced theory the vortices live on points, in the four-dimensional theory they live on two-dimensional sheets. Because of rotational invariance we must have $a = a_T$ and thus the formula for the reduced action follows.

Another way of formulating this condition is that the longitudinal measure on the size of vortices that we read from the classical action and the transverse measure on the vortices two-dimensional sheet must coincide.

If a quasi *BPS* Wilson loop in the adjoint representation is considered as in [1] vortices and anti-vortices contribute to the effective action and it is possible to pair the holomorphic integral to the anti-holomorphic one. In this case the counting of zero modes corresponds to the real dimension of the orbit. For completeness we write the local part of the effective action for a Wilson loop in the adjoint representation in the notation of [1]:

$$\begin{aligned}
 \exp(-\Gamma_q) &= \prod_p \sum_{k_p, e_p} \exp - \left(\frac{2\pi}{Ha^2} \frac{8\pi^2}{2g_W^2} Z^{-1} k_p (N - k_p) + c.c. \right) \\
 &\quad \exp - \left(\frac{2\pi}{H\tilde{a}^2} \frac{8\pi^2}{2g_W^2} (k_p (N - k_p) + N^2 e_p^2) + c.c. \right) \\
 &\quad \exp \left(\frac{2\pi}{Ha_T^2} \frac{k_p (N - k_p)}{2} \log \left(\frac{1}{Ha^2} \right) + c.c. \right) \\
 &\quad \exp \left(\frac{2\pi}{Ha_T^2} \frac{k_p (N - k_p)}{2} \log \left(\frac{1}{H\tilde{a}^2} \right) + c.c. \right)
 \end{aligned} \tag{4.14}$$

where e_p denotes the shift of the local curvature by a central large gauge transformation valued in Z_N .

The analogous expression for a Wilson loop in the fundamental representation is:

$$\begin{aligned}
 \exp(-\Gamma_q) &= \prod_p \sum_{k_p, n_p} \exp - \left(\frac{2\pi}{Ha^2} \frac{8\pi^2}{2g_W^2} Z^{-1} k_p (N - k_p) \right) \\
 &\quad \exp - \left(\frac{2\pi}{H\tilde{a}^2} \frac{8\pi^2}{2g_W^2} (k_p (N - k_p) + N^2 n_p^2) \right) \\
 &\quad \exp \left(\frac{2\pi}{Ha_T^2} \frac{k_p (N - k_p)}{2} \log \left(\frac{1}{Ha^2} \right) \right) \\
 &\quad \exp \left(\frac{2\pi}{Ha_T^2} \frac{k_p (N - k_p)}{2} \log \left(\frac{1}{H\tilde{a}^2} \right) \right)
 \end{aligned} \tag{4.15}$$

We now come to the canonical coupling. We start recalling how the difference between the Wilsonian and the canonical beta function can be understood in terms of a rescaling anomaly in the functional integral, in the $\mathcal{N} = 1$ supersymmetric case, following [49].

The canonical coupling constant can be related to the Wilsonian one taking into account an anomalous Jacobian that occurs in the functional integral:

$$\begin{aligned}
 Z &= \int \exp \left(-\frac{N}{2g_W^2} S_{YM}(A) \right) DA \\
 &= \int \exp \left(-\frac{N}{2g_W^2} S_{YM}(g_c A_c) \right) \frac{D(g_c A_c)}{DA_c} DA_c \\
 &= \int \exp \left(-\frac{N}{2g_W^2} S_{YM}(g_c A_c) + \log \frac{D(g_c A_c)}{DA_c} \right) DA_c \\
 &= \int \exp \left(-\frac{N}{2g_c^2} S_{YM}(g_c A_c) \right) DA_c
 \end{aligned} \tag{4.16}$$

From this relation it follows that:

$$\frac{N}{2g_c^2} = \frac{N}{2g_W^2} - S_{YM}^{-1}(g_c A_c) \log \frac{D(g_c A_c)}{DA_c} \tag{4.17}$$

We observe that the fields at the ultraviolet and at the infrared are not canonically normalized in the reduced *EK* effective action. It is however difficult to canonically normalize the reduced action because the only possibility would be to rescale the vortices eigenvalues. However they are strongly rigid since they are quantized and thus we have not this freedom. A way out is to use the unreduced effective action. In the unreduced effective action we can canonically normalize the fields at the ultraviolet and at the infrared by writing $H = \frac{Z^{-1}}{g_c^2} H_c$ and $\tilde{a}^2 = Z\tilde{a}_c^2$ and by keeping fixed these canonically defined scales. However because of rotational invariance also the canonically defined transverse cut-off must be kept fixed $a_T^2 = Z a_{Tc}^2$. Finally, we consistently set $a = a_{Tc}$, since the longitudinal measure on the size of vortices that we read from the classical canonical action and the transverse measure on the vortices two-dimensional sheet must coincide as in the Wilsonian case. This way of defining the canonical transverse cut-off a_{Tc} implies that the rescaled transverse measure does not depend on the rescaling factor being g_c^2 or $g_c^2 Z$, as it should be.

After a new *EK* reduction in which this time $N_2 = \frac{2\pi}{H_c a^2}$ and the unreduced action is divided by N_2 we get the following relation between the Wilsonian and the canonical coupling:

$$\frac{1}{2g_W^2} = \frac{1}{2g_c^2} + \beta_J \log g_c + \frac{\beta_J}{4} \log Z \tag{4.18}$$

Differentiating this relation we get for the canonical beta function [1]:

$$\frac{\partial g_c}{\partial \log \Lambda} = \frac{-\beta_0 g_c^3 + \frac{\beta_J}{4} g_c^3 \frac{\partial \log Z}{\partial \log \Lambda}}{1 - \beta_J g_c^2} \tag{4.19}$$

Now we come to the computation of the anomalous dimension. Since Z is one-loop exact it leads to the anomalous dimension:

$$\frac{\partial \log Z}{\partial \log a} = - \frac{\frac{1}{(4\pi)^2} \frac{10}{3} g_W^2}{1 - g_W^2 \frac{1}{(4\pi)^2} \frac{10}{3} \log\left(\frac{\tilde{a}}{\sqrt{N_D} a}\right)} \tag{4.20}$$

where now we have included the contribution of the conformal anomaly, that combines with the subtraction point to give a finite but arbitrary result for the higher order contributions to the anomalous dimension. Thus the *RG* trajectory must be followed along the line $c = -\frac{1}{(4\pi)^2} \frac{10}{3} \log\left(\frac{\tilde{a}}{\sqrt{N_D} a}\right) = \text{constant}$. We observe that it is precisely the contribution of the conformal anomaly that makes the anomalous dimension a function of the Wilsonian coupling only, according with the *RG*. We conclude that the loop equation and the beta function for the quasi *BPS* Wilson loops are saturated by the Z_N vortices of the partial *EK* reduction:

$$\begin{aligned} F_{z\bar{z}} + i[D_u, D_{\bar{u}}] &= i \left(\sum_p g_p \lambda_p g_p^{-1} \delta^{(2)}(z - z_p) - H1 \right) \\ \bar{\partial}_A(D_u) &= 0 \\ \partial_A(D_{\bar{u}}) &= 0 \end{aligned} \tag{4.21}$$

5 Physical effective charge

We exploit the residual scheme dependence left in our approach to find a link with the physical charge between static quark sources in the large- N limit. The scheme dependence arises because the conformal anomaly in the Wilsonian effective action is fine-tuned along the RG flow, in order to obtain a finite result for the higher order contributions to the anomalous dimension. These contributions are then finite but contain an arbitrary dimensionless parameter c . In turn the c dependence of the anomalous dimension determines the RG flow in the infrared of the canonical coupling.

We argue that there should exist in our approach a scheme in which the canonical coupling coincides with the physical effective charge, essentially because of the very definition of our canonical coupling via the loop equation.

By physical effective charge we mean the coefficient of the Coulomb-like potential in the complete inter-quark potential, after having separated the linear confining term proportional to the string tension, by taking second derivatives of the potential as in [23]:

$$V(r) = -\frac{g_{\text{phys}}^2(r)}{4\pi r} + Kr \quad (5.1)$$

where K is the string tension. With this definition the physical charge may still possess, from a stringy point of view, non-perturbative string-like contributions [50] proportional, in the large distance limit, to inverse powers of \sqrt{Kr} . In fact it is known from the string model for the large distance inter-quark potential [50] that the universal infrared asymptotic value of $\frac{g_{\text{phys}}^2}{4\pi}$, equal to $\frac{\pi}{12}$, might get non-universal corrections in powers of \sqrt{Kr} . These corrections in principle could start at $\frac{1}{\sqrt{Kr}}$ order but the analysis in [50] shows that, assuming open/closed string duality, they in fact could start only at order of $\frac{1}{(\sqrt{Kr})^2}$. Another important point is that the beta function of the physical charge thus defined may possess an infrared fixed point without implying that the large- N YM theory be conformal in the infrared, because the conformal invariance is always broken by the confining linear term in the inter-quark potential, that shows up in physical Wilson loops.

A direct gauge theoretic computation of the physical charge needs the evaluation of physical Wilson loops. In our approach only quasi BPS Wilson loops can be computed directly. However there are physical Wilson loops that can be obtained by analytic continuation from quasi BPS Wilson loops. This will be considered elsewhere [51]. Yet it is natural to assume, because of the universality of the renormalization procedure, that the canonical coupling that arises in the loop equation for the quasi BPS Wilson loop coincides in some scheme with the physical charge. Since the canonical coupling is a local quantity that is determined only by the local part of the effective action, while the physical charge may get contributions also from the non-local part, if it does not vanish in the large- N limit, we can hope to identify the canonical coupling in some scheme with the effective charge only in the local limit for the effective action, that is the limit in which the effective action is actually computed in this paper.

Thus we find that if the effective action in the large- N limit implies non-local interactions between the vortices on which it is localized, then these interactions might imply

the existence of $\frac{1}{\sqrt{K_r}}$ string-like corrections to the physical effective charge. Otherwise the physical effective charge is completely accounted by the canonical coupling constant in some scheme.

But then this scheme is uniquely determined, as we will see momentarily. We find that the effective charge has a (non-conformal) infrared fixed point at the inverse RG invariant scale in the Wilsonian scheme, i.e. at the Landau pole of the Wilsonian coupling.

Given our formulae for the beta function of large- N YM :

$$\frac{\partial g_c}{\partial \log \Lambda} = \frac{-\beta_0 g_c^3 + \frac{\beta_J}{4} g_c^3 \frac{\partial \log Z}{\partial \log \Lambda}}{1 - \beta_J g_c^2} \tag{5.2}$$

$$\frac{\partial \log Z}{\partial \log \Lambda} = \frac{\gamma_0 g_W^2}{1 + c g_W^2} \tag{5.3}$$

$$\gamma_0 = \frac{1}{(4\pi)^2} \frac{10}{3} \tag{5.4}$$

$$\frac{\partial g_W}{\partial \log \Lambda} = -\beta_0 g_W^3 \tag{5.5}$$

we inquire as to whether we can find a value of c for which the canonical coupling coincides with the gauge invariant physical charge between static quark sources in the large- N limit.

The scheme of the physical charge is uniquely fixed (up perhaps to the mentioned non-local contributions) by the requirement that the physical charge be continuous and differentiable. Indeed we can study the behavior of the RG flow as a function of c .

For c negative the anomalous dimension $\frac{\partial \log Z}{\partial \log \Lambda}$ has a pole at a finite value of g_W . Thus the canonical beta function has a zero in the numerator at a finite distance. However the derivative of the physical charge does not vanish at that zero. This means that the derivative is discontinuous, since on that point on the physical charge stays constant. Thus c negative is not acceptable in the scheme of the physical charge.

For c positive we must distinguish three cases. For $c > \frac{1}{(4\pi)^2} \frac{\gamma_0}{\beta_0}$ the RG flow from the ultraviolet ends into a cusp, that is an infrared fixed point at the value $g_c^2 = \frac{(4\pi)^2}{4}$, where the beta function has a pole as in the $\mathcal{N} = 1$ $SUSY$ YM theory. However a cusp is not acceptable as the end of the RG flow of the physical charge, because of the divergence of the derivative of the inter-quark potential.

For $\frac{1}{(4\pi)^2} \frac{\gamma_0}{\beta_0} > c$ the flow ends into an infrared fixed point, but the flow is continuously differentiable (with zero derivative) only at the critical value of $c = \frac{1}{(4\pi)^2} \frac{\gamma_0}{\beta_0}$. From that point on the canonical coupling remains constant in this scheme. The scale at which this occurs is Λ_W^{-1} , the inverse RG invariant scale in the Wilsonian scheme. We can write the formulae for the beta function in this scheme:

$$\frac{\partial g_{\text{phys}}}{\partial \log \Lambda} = -\beta_0 \frac{g_{\text{phys}}^3}{1 - \beta_J g_{\text{phys}}^2} \frac{1}{1 + \frac{1}{(4\pi)^2} \frac{10}{11} g_W^2} \tag{5.6}$$

$$\frac{\partial g_W}{\partial \log \Lambda} = -\beta_0 g_W^3 \tag{5.7}$$

Eliminating g_W as a function of Λ_W :

$$g_W^2 = -\frac{1}{2\beta_0 \log(r\Lambda_W)} \tag{5.8}$$

we get for the physical beta function as a function of the inter-quark distance:

$$\frac{\partial g_{\text{phys}}}{\partial \log r} = \beta_0 \frac{g_{\text{phys}}^3}{1 - \beta_J g_{\text{phys}}^2} \frac{\log(r\Lambda_W)}{\log(r\Lambda_W) - \frac{15}{121}} \tag{5.9}$$

It is then easy to see that the junction point with the constant flow is C^∞ and not only continuous and differentiable. Since YM theory contains only one arbitrary parameter, in our case Λ_W , the value of the physical charge at the infrared fixed point is not arbitrary. However it cannot be found using only the formula for the beta function, but it needs a direct computation from physical Wilson loops. It is possible to perform this computation by analytic continuation from quasi BPS Wilson loops to physical Wilson loops [51] and by our Wilsonian effective action (the infrared fixed point of the canonical coupling is the Landau pole of the Wilsonian coupling).

Finally we should observe that the prediction of the existence of an infrared fixed point in the large- N limit for our definition of the physical charge compares favorably, within the expected order of $\frac{1}{N}$ accuracy, with the plateau observed in [23] for the group $SU(3)$. Of course a large- N lattice computation would be more than welcome.

6 Conclusions

We have shown the existence in the large- N limit of pure YM and of $\mathcal{N} = 1$ $SUSY$ YM of quasi BPS Wilson loops that are protected by some of the usual renormalizations, since they have no perimeter and no cusp divergences for backtracking cusps, as in cases with extended $SUSY$.

The existence of such objects is used to localize a version of the loop equation in the ASD variables on a saddle-point equation for an effective action. The proof of localization is obtained by homological methods as opposed to the cohomological localization in local field theory.

The crucial point is that the loop equation reduces to the insertion of the equation of motion for an effective action once the local degrees of freedom of the theory are holographically mapped by a local conformal transformation into backtracking arcs ending into cusps at infinity.

This effective action contains the whole information about the only remaining renormalization of quasi BPS Wilson loops, i.e. charge renormalization.

An explicit formula for the canonical beta function of large- N pure YM theory follows.

At the same time this shows the existence [52] of the large- N limit of the pure YM theory for this (almost trivial, at least in perturbation theory) class of observables.

However one key point is that, since charge renormalization must be the same in all the sectors of the theory, essentially by the same token we get information on the physical charge between static quark sources in the large- N limit.

For the future the obvious interesting new game consists in trying to extend these techniques to those physical Wilson loops that can be obtained by analytic continuation of quasi *BPS* Wilson loops.

Another interesting aside consists in reproducing the known beta function in the $\mathcal{N} = 1$ *SUSY* case by the same technique.

Finally the present results suggest, in the light of the role that topological strings play in the localization of the loop equation on a saddle-point, that a purely stringy approach to the solution of the loop equation should exist, according to a long-standing conjecture. In particular the localization on non-commutative vortices suggests that the equations of motion in target space of the string theory dual to large- N *YM* should be the vortices equations of *ASD* type in presence of a *B*-field.

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